Random periodic solutions for McKean-Vlasov SDEs

Jianhai Bao

Tianjin University

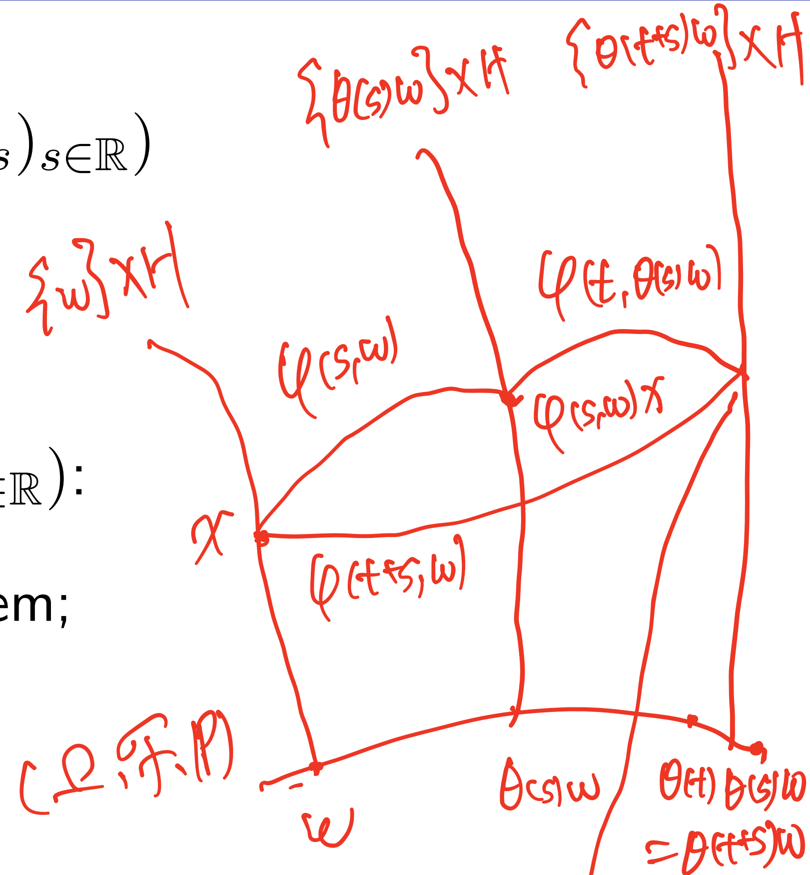
Outline

- Motivations
- RPSs for SDEs: partially dissipative
- RPSs for functional SDEs: dissipative on average
- RPSs for McKean-Vlasov SDEs: dissipative on average
- RPSs for McKean-Vlasov SDEs: partially dissipative

Random dynamical system

- Measurable dynamical system: $(\Omega, \mathscr{F}, (\theta_s)_{s\in\mathbb{R}})$
  - $(\omega, t) \mapsto \theta_t\omega$: measurable;
  - $\theta_0 = \mathrm{id}$, $\theta_{s+t} = \theta_s \circ \theta_t$.
- Metric dynamical system $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_s)_{s\in\mathbb{R}})$:
  - $(\Omega, \mathscr{F}, (\theta_s)_{s\in\mathbb{R}})$: metric dynamical system;
  - $(\theta_s)_{s\in\mathbb{R}}$: measure preserving.
- Random dynamical system:
  - $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_s)_{s\in\mathbb{R}})$: metric dynamical system;
  - $\psi : \mathbb{R} \times \Omega \times H \to H$: measurable; identity; cocycle property,

$$\psi(t+s,\omega)x = \psi(t, \theta_s\omega) \circ \psi(s,\omega)x.$$



Random periodic solution

- Pathwise random τ-periodic solution
  - $u : \triangle \times \Omega \times H \to H$: semi-flow;
  - $y : \mathbb{R} \times \Omega \to H$: $\mathscr{F}$-measurable s.t.

$$u(t, s, \omega, y(s, \omega)) = y(t, \omega), \qquad y(s + \tau, \omega) = y(s, \theta_\tau \omega).$$

- Random τ-periodic solution in distribution
  - $u : \triangle \times \Omega \times H \to H$: semi-flow;
  - $y : \mathbb{R} \times \Omega \to H$: $\mathscr{F}$-measurable s.t.

$$u(t, s, \omega, y(s, \omega)) \stackrel{d}{=} y(t, \omega), \qquad y(s + \tau, \omega) \stackrel{d}{=} y(s, \theta_\tau \omega).$$

Existing literature

Existing literature: time-inhomogeneous SDEs/SPDEs

- Feng-Zhao-Zhou'11: semi-linear SDE + additive noise;
- Feng-Zhao'11: semi-linear SPDE + bounded domain;
- Feng-Wu-Zhao'16: semi-linear SDE + multiplicative noise;

Potential methods:

- Pull-back approach: uniformly dissipative;
- Generalized Schauder's fixed point theorem: partially dissipative.

Remarks:

- Non-uniqueness;
- Leading linear term.

Motivating examples

- Drift $b_t(x) := \alpha_t(x - x^3), x \in \mathbb{R}$.

Uniform dissipativity in existing literature:

$$\langle x - y, b_t(x) - b_t(y)\rangle \leq -\lambda|x - y|^2.$$

Our concerns:

- Reinforce the periodicity:

$$\langle x - y, b_t(x - b_t(y))\rangle \leq \lambda(t)|x - y|^2.$$

- Partial dissipativity:

$$\langle x - y, b_t(x) - b_t(y)\rangle \leq \alpha_t\big((K_1 + K_2)\mathbf{1}_{\{|x-y|\leq L\}} - K_2\big)|x - y|^2.$$

SDEs: partially dissipative

SDE with additive noise:

$$\mathrm{d}X(t) = b_t(X(t))\,\mathrm{d}t + \sqrt{\alpha_t}\,\mathrm{d}W(t), \tag{1}$$

where

- $b : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d, \qquad \alpha : \mathbb{R} \to (0, \infty)$;
- $b(\cdot, x)$ and $\alpha(\cdot)$: τ-periodic and continuous;
- $\exists\ K_1, L \geq 0, K_2 > 0$ s.t.

$$\langle x - y, b_t(x) - b_t(y) \rangle \leq \alpha(t)\big((K_1 + K_2)\mathbf{1}_{\{|x-y| \leq L\}} - K_2\big)|x - y|^2.$$

Reference: Ren, Sturm and Wang'22, arXiv:2110.06473.

RPS in distribution: partially dissipative

Theorem

Under $(\mathbf{A})$, *the solution flow* ϕ *has a unique random* τ*-periodic solution in the sense of law, i.e.,* $\exists$ *a unique* $X^*(t) \in L^1(\Omega \to \mathbb{R}^d, \mathscr{F}_t, \mathbb{P})$ *such that*

$$X^*(t+h,\omega) \overset{d}{=} \phi(t+h,t,X^*(t,\omega),\omega), \qquad X^*(t+\tau,\omega) \overset{d}{=} X^*(t,\theta_\tau\omega),$$

and

$$\lim_{s\downarrow-\infty} X^{s,\xi}(t) \overset{d}{=} X^*(t).$$

Remarks

- Uniform dissipativity: $K_1 = L = 0$;
- The drift b is partial dissipative, e.g., $b_t(x) = \alpha_t(x - x^3)$;
- Uniform dissipativity: synchronous coupling;
- Our approach: reflection coupling;
- Random periodic solution: weak existence and uniqueness.

Remarks

- SDE with multiplicative noise:

$$\mathrm{d}X(t) = b(t, X(t))\,\mathrm{d}t + \sigma(t, X(t))\,\mathrm{d}W(t).$$

- $\exists\ \sigma_0 : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ s.t. $(\sigma\sigma^*)(t,x) = \alpha(t) I_{d\times d} + (\sigma_0\sigma_0^*)(t,x)$.
- $K_1, L \geq 0, K_2 > 0$ s.t.

$$\langle x-y, b(t,x)-b(t,y)\rangle + \frac{1}{2}\|\sigma_0(t,x)-\sigma_0(t,y)\|_{\mathrm{HS}}^2$$
$$\leq \alpha(t)\Big((K_1+K_2)\mathbf{1}_{\{|x-y|\leq L\}} - K_2\Big)|x-y|^2.$$

- Additive noise: reflection; multiplicative noise: synchronous.

Reference: Priola & Wang'06, JFA.

A general criteria on RPS in distribution

For an $\mathbb{R}^d$-valued Markov process $(Y^{s,x}(t))_{t\geq s}$,

- $(Y^{s,\xi}(t))_{t\geq s}$ enjoys the semi-flow property.
- $\exists\ C_0(\xi) > 0$ s.t.

$$\sup_{t\geq s} \mathbb{W}_\psi\big(\mathscr{L}_{Y^{s,\xi}(t)}, \delta_{\mathbf{0}}\big) \leq C_0(\xi).$$

- $\exists\ h : [0,\infty) \to [0,\infty)$ satisfying

$$\lim_{s\to-\infty} \sum_{j=0}^{\infty} h(t-s+j\tau) = 0,$$

and

$$\mathbb{W}_\psi\big(\mathscr{L}_{Y^{s,\xi}(t)}, \mathscr{L}_{Y^{s,\eta}(t)}\big) \leq h(t-s)\mathbb{W}_\psi\big(\mathscr{L}_\xi, \mathscr{L}_\eta\big).$$

A general criteria on RPS in distribution

Then, if $\phi : \triangle \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ defined via

$$\phi(t,s,x,\omega) = Y^{s,x}(t,\omega), \quad (t,s) \in \triangle,\ x \in \mathbb{R}^d,\ \omega \in \Omega,$$

further satisfies the following property:

$$\phi(t+\tau, s+\tau, x, \omega) = \phi(t,s,x,\theta_\tau \omega),$$

$\exists$ a unique (in the sense of law) $\mathscr{F}_t$-measurable $Y^*(t)$ such that

$$Y^*(t+h,\omega) \overset{\mathbb{W}_\psi}{=} \phi(t+h,t,Y^*(t,\omega),\omega), \qquad Y^*(t+\tau,\omega) \overset{\mathbb{W}_\psi}{=} Y^*(t,\theta_\tau\omega)$$

and

$$\lim_{s\downarrow -\infty} Y^{s,\xi}(t) \overset{\mathbb{W}_\psi}{=} Y^*(t).$$

Functional SDEs: dissipative on average

Functional SDE:

$$\mathrm{d}X(t) = b(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W(t), \qquad t \ge s \in \mathbb{R},$$

where

- $b(\cdot, \xi)$ and $\sigma(\cdot, \xi)$ are τ-periodic and continuous on $\mathbb{R}$;
- b: uniformly bounded on each bounded set of $\mathbb{R} \times \mathscr{C}$;
- $\exists$ continuous τ-periodic functions $\lambda_1 : \mathbb{R} \to \mathbb{R}, \lambda_2, \lambda_3 : \mathbb{R} \to [0, \infty)$ s.t.

$$2\langle b(t,\xi) - b(t,\eta), \xi(0) - \eta(0)\rangle \le \lambda_1(t)|\xi(0) - \eta(0)|^2 + \lambda_2(t)\|\xi - \eta\|_\infty^2,$$

$$\|\sigma(t,\xi) - \sigma(t,\eta)\|_{\mathrm{HS}}^2 \le \lambda_3(t)\|\xi - \eta\|_\infty^2.$$

Theorem

Assume $(\mathbf{H})$ *and*

$$\ell := \int_0^\tau \Big(\lambda_1(r) + 2\mathrm{e}^{-c_*(r_0,\tau)}\big(\lambda_2(r) + \lambda_3(r) + 2\lambda_3(r)\chi^2\mathrm{e}^{-c_*(r_0,\tau)+2c^*(r_0,\tau)}\big)\Big)\,\mathrm{d}r < 0,$$

where $\chi \approx 1.30693$ *and*

$$c_*(r_0,\tau) := \inf_{0\le u\le\tau,-r_0\le\theta\le 0}\int_{u+\theta}^{u}\lambda_1(s)\,\mathrm{d}s,$$

$$c^*(r_0,\tau) := \sup_{0\le u\le\tau,-r_0\le\theta\le 0}\int_{u+\theta}^{u}\lambda_1(s)\,\mathrm{d}s.$$

Then, the flow ϕ *has a a unique pathwise random* τ*-periodic solution.*

Remarks

- Non-dissipativity at some time points;
- Dissipativity on average;
- Reflection coupling: no longer work;
- Approach: synchronous coupling.

A general criteria on pathwise RPSs

For a Markov process $(Y^{s,x}(t))_{t\geq s}$ on $(\mathbb{U},\rho)$ and some $p>0$,

- $(Y^{s,\xi}(t))_{t\geq s}$ enjoys the semi-flow property;
- $\exists\ C_0(\xi)>0$ such that

$$\sup_{t\geq s}\mathbb{E}\rho(Y^{s,\xi}(t),\mathbf{0})^p\leq C_0(\xi).$$

- $\exists\ h:[0,\infty)\to[0,\infty)$ satisfying

$$\lim_{s\to-\infty}\sum_{j=0}^{\infty}h(t-s+j\tau_0)=0$$

and s.t.

$$\Big(\mathbb{E}\rho\big(Y^{s,\xi}(t),Y^{s,\eta}(t)\big)^p\Big)^{\frac{1}{1\vee p}}\leq h(t-s)\Big(\mathbb{E}\rho(\xi,\eta)^p\Big)^{\frac{1}{1\vee p}}.$$

A general criteria on pathwise RPS

Then, if the map $\phi : \triangle \times \mathbb{U} \times \Omega \to \mathbb{U}$ defined via

$$\phi(t, s, x, \omega) = Y^{s,x}(t, \omega)$$

further satisfies the following property:

$$\phi(t + \tau, s + \tau, x, \omega) = \phi(t, s, x, \theta_\tau \omega),$$

there exists a unique $\mathscr{F}_t$-measurable $Y^*(t)$ such that

$$Y^*(t+r, \omega) = \phi(t+r, t, Y^*(t, \omega), \omega), \qquad Y^*(t+\tau, \omega) = Y^*(t, \theta_\tau \omega) \quad \text{a.s.}$$

and moreover

$$\lim_{s \downarrow -\infty} \mathbb{E}\rho(Y^{s,\xi}(t), Y^*(t))^p = 0.$$

McKean-Vlasov SDE: dissipative on average

McKean-Vlasov SDE:

$$\mathrm{d}X_{s,t} = b_t(X_{s,t}, \mathscr{L}_{X_{s,t}})\,\mathrm{d}t + \sigma_t(X_{s,t}, \mathscr{L}_{X_{s,t}})\,\mathrm{d}W_t.$$

Assume

(A) $\mathbb{R} \ni t \mapsto |b_t(\mathbf{0}, \delta_{\mathbf{0}})|^2 + \|\sigma_t(\mathbf{0}, \delta_{\mathbf{0}})\|_{\mathrm{HS}}^2$ is continuous, and b_t is continuous and bounded on bounded sets of $\mathbb{R}^d \times \mathscr{P}_2(\mathbb{R}^d)$. Moreover, $\exists\, \overline{K}_1, \overline{K}_2 \in L_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}_+)$ such that

$$2\langle x - y, b_t(x,\mu) - b_t(y,\nu)\rangle \le \overline{K}_1(t)\big(|x-y|^2 + \mathbb{W}_2(\mu,\nu)^2\big),$$

and $\|\sigma_t(x,\mu) - \sigma_t(y,\nu)\|_{\mathrm{HS}}^2 \le \overline{K}_2(t)\big(|x-y|^2 + \mathbb{W}_2(\mu,\nu)^2\big)$.

McKean-Vlasov SDE: dissipative on average

(A') Assume **(A)**. In addition, $t \mapsto b_t(x,\mu)$ and $t \mapsto \sigma_t(x,\mu)$ are τ-periodic. Moreover, $\exists\ K_3 \in C(\mathbb{R};\mathbb{R}_+)$ and $K_1, K_2 \in C(\mathbb{R};\mathbb{R})$ such that

$$
\begin{aligned}
&2\langle x-y, b_t(x,\mu)-b_t(y,\nu)\rangle + \|\sigma_t(x,\mu)-\sigma_t(y,\nu)\|_{\mathrm{HS}}^2 \\
&\le K_1(t)|x-y|^2 + K_2(t)\mathbb{W}_2(\mu,\nu)^2,
\end{aligned}
$$

and

$$
\|\sigma_t(x,\mu)-\sigma_t(y,\nu)\|_{\mathrm{HS}}^2 \le K_3(t)\big(|x-y|^2 + \mathbb{W}_2(\mu,\nu)^2\big).
$$

Pathwise RPS: dissipative on average

Theorem

Assume $(\mathbf{A}')$ *with* $-\lambda := \int_0^\tau \big(K_1(u) + K_2(u)\big)\,\mathrm{d}u < 0$. *Then, the solution flow possesses a unique pathwise random* τ*-periodic solution, i.e.,* $\exists$ *a unique* $X_t^* \in L^2(\Omega \to \mathbb{R}^d, \mathscr{F}_t, \mathbb{P})$ *such that*

$$X_{t+h}^*(\omega) = \phi(t+h, t, X_t^*(\omega), \omega), \qquad X_{t+\tau}^*(\omega) = X_t^*(\theta_\tau \omega) \quad \text{a.s.}$$

Moreover,

$$\lim_{s \downarrow -\infty} \mathbb{E}|X_{s,t}^{\xi} - X_t^*|^2 = 0.$$

Pathwise RPS: dissipative on average

Theorem

Under Assumptions of Theorem 3, the solution flow ϕ^N *possesses a unique pathwise random* τ*-periodic solution, i.e.,* $\exists$ *a unique* $\mathbf{X}_t^{*,N} \in L^2(\Omega \to \mathbb{R}^{dN}, \mathscr{F}_t, \mathbb{P})$ *such that*

$$\mathbf{X}_{t+h}^{*,N}(\omega) = \phi^N(t+h, t, \mathbf{X}_t^{*,N}(\omega), \omega), \qquad \mathbf{X}_{t+\tau}^{*,N}(\omega) = \mathbf{X}_t^{*,N}(\theta_\tau \omega) \quad \text{a.s.}$$

Furthermore,

$$\lim_{s \downarrow -\infty} \mathbb{E}|\mathbf{X}_{s,t}^{\xi^N} - \mathbf{X}_t^{*,N}|^2 = 0.$$

Unform propagation of chaos: dissipative on average

Theorem

Assume Assumptions of Theorem 3. Then, $\exists\ C > 0$ *such that*

$$\mathbb{E}|X^i_{s,t} - X^{i,N}_{s,t}|^2 + \mathbb{E}|X^{i,*}_t - X^{i,*,N}_t|^2 \le C\varphi(N) \to 0 \quad as \quad N \to \infty,$$

McKean-Vlasov SDE: partially dissipative

McKean-Vlasov SDE:

$$\mathrm{d}X_{s,t} = \big(\widehat{b}_t(X_{s,t}) + (\widetilde{b}_t * \mathscr{L}_{X_{s,t}})(X_{s,t})\big)\,\mathrm{d}t + \sqrt{\alpha_t}\mathrm{d}B_t + \widehat{\sigma}_t(X_{s,t})\,\mathrm{d}W_t.$$

Assume

(H) Fix $\tau \in (0, \infty)$. Let $\mathbb{R} \ni t \mapsto b_t(\mathbf{0})$, $\mathbb{R} \ni t \mapsto \widehat{\sigma}_t(\mathbf{0})$, $\mathbb{R} \ni t \mapsto \widetilde{b}_t(\mathbf{0}, \mathbf{0})$ and $\mathbb{R} \ni t \mapsto \alpha_t$ are τ-periodic and $\exists\ K_0, \ell_0 \geq 0, K_1, K_2, K_3 > 0$ s.t.

$$\begin{aligned}\langle x - y, \widehat{b}_t(x) - \widehat{b}_t(y)\rangle &+ \frac{1}{2}\|\widehat{\sigma}_t(x) - \widehat{\sigma}_t(y)\|_{\mathrm{HS}}^2 \\ &\leq \alpha_t\big((K_0 + K_1)\mathbf{1}_{\{|x-y|\leq \ell_0\}} - K_1\big)|x - y|^2,\end{aligned}$$

and

$$|\widetilde{b}_t(x, y) - \widetilde{b}_t(\widetilde{x}, \widetilde{y})| \leq K_2\alpha_t\big(|x - \widetilde{x}| + |y - \widetilde{y}|\big),$$

$$\|\widehat{\sigma}_t(x) - \widehat{\sigma}_t(y)\|_{\mathrm{HS}}^2 \leq K_3\alpha_t|x - y|^2.$$

RPS in distribution: partially dissipative

Theorem

Under $(\mathbf{H})$, $\exists\ K_2^* > 0$ *such that, for all* $K_2 \in (0, K_2^*]$, *the solution flow* ϕ *has a unique random* τ*-periodic solution in the sense of distribution, i.e.,* $\exists$ *a unique* $X_t^* \in L^1(\Omega \to \mathbb{R}^d, \mathscr{F}_t, \mathbb{P})$ *such that*

$$X_{t+h}^*(\omega) \stackrel{d}{=} \phi(t+h, t, X_t^*(\omega), \omega), \qquad X_{t+\tau}^*(\omega) \stackrel{d}{=} X_t^*(\theta_\tau \omega).$$

Moreover,

$$\lim_{s \downarrow -\infty} X_{s,t}^{\xi} \stackrel{d}{=} X_t^*.$$

RPS in distribution: partially dissipative

Theorem

Under $(\mathbf{H})$, $\exists\ K_2^* > 0$ *such that, for all* $K_2 \in (0, K_2^*]$ *and* $N \geq 1$, *the solution flow* ϕ^N *admits a unique random* τ*-periodic solution in the sense of distribution, i.e.,* $\exists$ *a unique* $\mathbf{X}_t^{*,N} \in L^2(\Omega \to \mathbb{R}^{dN}, \mathscr{F}_t, \mathbb{P})$ *such that*

$$\mathbf{X}_{t+h}^{*,N}(\omega) \stackrel{d}{=} \phi(t+h, t, \mathbf{X}_t^{*,N}(\omega), \omega), \qquad \mathbf{X}_{t+\tau}^{*,N}(\omega) \stackrel{d}{=} \mathbf{X}_t^{*,N}(\theta_\tau \omega).$$

Moreover, $\lim_{s \downarrow -\infty} \mathbf{X}_{s,t}^{\xi^N} \stackrel{d}{=} \mathbf{X}_t^{*,N}$.

Theorem

Assume Assumptions of Theorem 7. Then, $\lim_{N \to \infty} X_t^{i,*,N} \stackrel{d}{=} X_t^{i,*}$.

Uniform Propagation of chaos: partially dissipative

Theorem

Under $(\mathbf{H})$, $\exists\ C_1, C_2, \lambda, K_2^* > 0$ *such that for all* $K_2 \in (0, K_2^*]$,

$$\mathbb{W}_1\left(\mathscr{L}_{X^i_{s,t}}, \mathscr{L}_{X^{i,N}_{s,t}}\right) \le C_1 \mathrm{e}^{-\lambda(t-s)} \mathbb{W}_1\left(\mathscr{L}_{X^i_{s,s}}, \mathscr{L}_{X^{i,N}_{s,s}}\right) + \frac{C_2}{\sqrt{N}}.$$

Tool: Reflection coupling.